ARHANGEL'SKIĬ SHEAF AMALGAMATIONS IN TOPOLOGICAL GROUPS (INTERMEDIATE REPORT)

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Abstract. The main results of this note are:

(1) For arbitrary topological groups, Nyikos's property $\alpha_{1.5}$ is equivalent to Arhangel'skiĭ's α_1.
(2) There is a topological space X such that $C_p(X)$ is α_1 but is not countably tight.

Item (1) solves a problem of Shakhmatov (2002), and is proved using a new kind of perturbation argument. Item (2), which is proved by quoting known results (of Arhangel'skiĭ-Pytkeev, Moore and Todorčević), gives a new solution, with remarkable properties, to a problem of Averbukh and Smolyanov (1968) concerning topological vector spaces. This problem was first solved by Plichko (2009) using Banach spaces with weaker locally convex topologies.

Dedicated to Dikran Dikranjan, on the occasion of his 60*-th birthday*

1. Sheaf amalgamations in topological groups

To avoid trivialities, by *convergent sequence* $x_n \to x$ we mean a proper one, that is, such that $x \neq x_n$ for all n. This way, convergence is a property of countably infinite sets: A countably infinite set A converges to x if all (equivalently, some) bijective enumerations of A converge to x. Thus, in the following, by *sequence* we always mean a countably infinite set. The following concepts are due to Arhangel'skiĭ [1, 2], except for $\alpha_{1.5}$ which is due to Nyikos [10].

Definition 1.1. A topological space X is α_i, $i = 1, 1.5, 2, 3, 4$, if, respectively, for each $x \in X$ and all pairwise disjoint sequences $S_1, S_2, \ldots \subseteq X$, each converging to x, there is a sequence $S \subseteq \bigcup_n S_n$ such that S converges to x, and

(α_1) $S_n \setminus S$ is finite for all n.
($\alpha_{1.5}$) $S_n \setminus S$ is finite for infinitely many n.
(α_2) $S_n \cap S$ is infinite for all n.
(α_3) $S_n \cap S$ is infinite for infinitely many n.
(α_4) $S_n \cap S$ is nonempty for infinitely many n.

In the integer-indexed properties α_i, we may remove the requirement that the sequences $S_1, S_2, \ldots$ are pairwise disjoint [10]. Indeed, we can move to $S'_n = S_n \setminus \bigcup_{k<n} S_k$, $n \in \mathbb{N}$. If S'_n is infinite for infinitely many n, we can dispose of the other ones. Otherwise, $S = \bigcup_{k<n} S_k$ for some n with S'_n finite would be as required in (α_1). However, removing the disjointness requirement from $\alpha_{1.5}$ renders it superfluous: Applying it to the modified sequence $\bigcup_{k\leq n} S_n$, $n \in \mathbb{N}$, the obtained S would be as in (α_1).

Each of the properties in Definition 1.1 implies the subsequent one. To see that $\alpha_{1.5}$ implies α_2, for each n decompose $S_n = \bigcup_k S_{nk}$, and take $S'_n = \bigcup_{m\leq n} S_{mn}$ [10]. A survey of these properties is available in [16].

None of the implications

$$\alpha_1 \Rightarrow \alpha_{1.5} \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_1$$

can be (provably) reversed. Not even in the class of Fréchet-Urysohn spaces [16]. Recall that a topological space X is *Fréchet-Urysohn* if each point in the closure of a set is in fact a limit of a sequence in that set.

In the present paper, we consider these properties in the context of *topological groups*. This direction was pioneered by Nyikos in his 1981 paper [9], where he proved, among other things, that Fréchet-Urysohn groups are α_4, and that sequential α_2 groups are Fréchet-Urysohn. Shakhmatov [15] constructed, in the Cohen reals model, an example of a Fréchet-Urysohn group which is not α_3, and a Fréchet-Urysohn α_2 group which is not $\alpha_{1.5}$. In particular, none of the implications

$$\alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_1$$

is provably reversible in the realm of topological groups. The question whether $\alpha_{1.5}$ groups are α_1 is implicit in Shakhmatov's paper. The problem whether *Fréchet-Urysohn* $\alpha_{1.5}$ groups are α_1 is stated there. This variant of the problem was settled in the positive by Shibakov, in his 1999 paper [17].

In his 2002 survey chapter for *Recent Progress in Topology* [16], Shakhmatov cites Shibakov's solution, and writes: "It seems unclear if $\alpha_{1.5}$ and α_1 are equivalent for all (i.e., not necessarily Fréchet-Urysohn) topological groups." For groups of the form $C_p(X)$, the continuous real-valued functions on a space X, with the topology of pointwise convergence, Sakai answered this problem in the positive [12]. One step in his solution, uses a trick which was used earlier by Scheepers [14] to show that for $C_p(X)$, $\alpha_2 = \alpha_3 = \alpha_4$: Replace the n-th sequence $\{f_{nm} : m \in \mathbb{N}\}$ by $\{|f_{1m}| + \cdots + |f_{nm}| : m \in \mathbb{N}\}$. This approach is not applicable to arbitrary topological groups. Indeed, Sakai proves some of his lemmata in the context of general topological groups, but his main theorems are proved only in the case of $C_p(X)$.

Our main result is that, for all topological groups, $\alpha_{1.5}$ implies α_1.

Theorem 1.2. *A topological group is $\alpha_{1.5}$ if, and only if, it is α_1.*

Proof. Let G be a topological group, and $S_1, S_2, \ldots \subseteq G$ be sequences converging to e. Let T be any sequence converging to e (for example, take $T = S_1$). For each n, fix a bijective enumeration $S_n = \{g_{nm} : m \in \mathbb{N}\}$.

Let $\{(n_k, m_k) : k \in \mathbb{N}\}$ be an enumeration of $\mathbb{N} \times \mathbb{N}$ where each (n, m) appears infinitely often. For each k, as $(T \setminus \{t_1, \ldots, t_{k-1}\}) \cdot g_{n_k m_k}$ is infinite, we can pick

$$t_k \in T \setminus \{t_1, \ldots, t_{k-1}\}$$

such that

$$t_k g_{n_k m_k} \notin \{t_1 g_{n_1 m_1},\ \ldots\ ,\ t_{k-1} g_{n_{k-1} m_{k-1}}\}.$$

For each pair (n, m), let $\{k(n, m, i) : i \in \mathbb{N}\}$ be an increasing enumeration of $\{k : (n_k, m_k) = (n, m)\}$. Note that $(n, m, i) \mapsto k(n, m, i)$ is injective. For each i, define the following perturbation of S_n:

$$S_n^{(i)} = \{t_{k(n,1,i)} g_{n1},\ t_{k(n,2,i)} g_{n2},\ t_{k(n,3,i)} g_{n3},\ \ldots\}.$$

By the construction, the sets $S_n^{(i)}$, $n, i \in \mathbb{N}$ are pairwise disjoint, and therefore so are the sets
$$S'_n = S_1^{(n)} \cup S_2^{(n)} \cup \cdots \cup S_n^{(n)},$$
$n \in \mathbb{N}$.

Apply $\alpha_{1.5}$ to $S'_1, S'_2, \ldots$, to find a sequence S' converging to e, such that $S'_n \setminus S'$ is finite for each n in an infinite set $I \subseteq \mathbb{N}$. Define
$$S = \bigcup_{n\in I} \bigcup_{j=1}^{n} \{g_{jm} : m \in \mathbb{N}, t_{k(j,m,n)} g_{jm} \in S'\}.$$
Since for each $n \in I$ and each $j = 1, \ldots, n$, $t_{k(j,m,n)} g_{jm} \in S'$ for all but finitely many m, we have that $S_j \setminus S$ is finite for all j.

Finally, note that S is obtained by taking a subsequence of S' and multiplying its elements by distinct elements $t^{-1}_{k(j,m,n)}$, that is elements of a subsequence of $\{t^{-1} : t \in T\}$, which also converges to e. Thus, S converges to e, too. □

As an application, we give a very short proof of a result of Nogura and Shakhmatov.

Definition 1.3 (Nogura-Shakhmatov [8])**.** A topological space X is *Ramsey* if, whenever $\lim_n \lim_m x_{nm} = x$, there is an infinite $I \subseteq \mathbb{N}$ such that for each neighborhood U of x, there is k such that $\{x_{nm} : k < n < m,\ n, m \in I\} \subseteq U$.

In the context of topological groups, this definition simplifies to the following one.

Lemma 1.4 (Sakai [12])**.** *A topological group G is Ramsey if, and only if, whenever $\lim_m g_{nm} = e$ for all n, there is an infinite $I \subseteq \mathbb{N}$ such that $\{g_{nm} : n, m \in I, n < m\}$ converges to e.*

Proof. Assume that $\lim_m g_{nm} = g_n$ and $\lim_n g_n = e$. For each n, define $g'_{nm} = g_n^{-1} g_{nm}$. Then $\lim_m g_{nm} = e$ for all n. □

Theorem 1.5 (Nogura-Shakhmatov [8])**.** *Every $\alpha_{1.5}$ topological group is Ramsey.*

Proof. Let G be an $\alpha_{1.5}$ topological group. By Theorem 1.2, G is α_1. Thus, there is an increasing $f : \mathbb{N} \to \mathbb{N}$ such that $\{x_{nm} : m \geq f(n)\}$ converges to e. Take I to be the image of f. □

2. New amalgamations

The following problem remains open.

Problem 2.1 (Shakhmatov [16])**.**

(1) *Is every (Fréchet-Urysohn) α_2 topological group Ramsey?*
(2) *Is every (Fréchet-Urysohn) Ramsey topological group is α_2?*

Sakai proved that for groups of the form $C_p(X)$, Ramsey is equivalent to α_2 [12]. The proof uses the "pullback" argument from Scheepers's proof, described above.

We define several new local properties, all related to Ramsey and α_2, and prove implications among them. Since we intend to consider our properties in the realm of topological groups, we do not define separate versions where only one point in the space is considered. This may be interesting for studies in more general contexts, which are not conducted here.

Definition 2.2. A topological space X is *locally Ramsey* if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all n, there is an infinite $I \subseteq \mathbb{N}$ such that $\{x_{nm} : n, m \in I, n < m\}$ converges to x.

Thus, locally Ramsey spaces are α_3. By Lemma 1.4, a topological group is Ramsey if, and only if, it is locally Ramsey.

Definition 2.3. A topological space X is α_{2^-} if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all $n \in \mathbb{N}$, there are $m_1 < m_2 < \dots$ such that $\bigcup_n \{x_{1m_n}, \dots, x_{nm_n}\}$ converges to x.

Corollary 2.4. *Every α_{2^-} topological space is α_2.* $\square$

Proposition 2.5.

1. *Every α_{2^-} topological space is locally Ramsey.*
2. *Every α_{2^-} topological group is Ramsey.*

Proof. (1) Take $m_1 < m_2 < \dots$ as in the definition of α_{2^-}, and set $I = \{m_n : n \in \mathbb{N}\}$.

(2) By (1) and Lemma 1.4. $\square$

Definition 2.6. A topological space X is α_{3^-} if, for each $x \in X$, whenever $\lim_m x_{nm} = x$ for all n, there are infinite $I, J \subseteq \mathbb{N}$ such that $\{x_{nm} : n \in I, m \in J, n < m\}$ converges to x.

Corollary 2.7. *Every locally Ramsey space is α_{3^-}, and every α_{3^-} space is α_3.* $\square$

The above-mentioned results of Scheepers and Sakai follow.

Corollary 2.8. *For topological groups of the form $C_p(X)$, the properties*

$$\alpha_{2^-}, \alpha_2, \alpha_{3^-}, \alpha_3, \alpha_4, \textit{ locally Ramsay, and Ramsey,}$$

are all equivalent.

Proof. By the above results, it suffices to show that α_4 implies α_{2^-} for such spaces. This follows Scheepers's argument: Given sequences $S_n = \{f_{nm} : m \in \mathbb{N}\}$ each converging to 0, replace each S_n with

$$S'_n = \{|f_{1m}| + \dots + |f_{nm}| : m \geq n\}.$$

Applying α_4 and thinning out, we obtain an increasing sequence of indices $m_1 < m_2 < \dots$, such that

$$|f_{1m_n}| + \dots + |f_{nm_n}|$$

converges to 0. Then $\bigcup_n \{f_{1m_n}, \dots, f_{nm_n}\}$ converges to 0. $\square$

Definition 2.9. Let X be a topological space, and $x \in X$. The game $\alpha_2^{\mathrm{game}}(X, x)$ is played by two players, ONE and TWO, and has an inning per each natural number. On the nth inning, ONE chooses a sequence S_n converging to x, and TWO responds by choosing an infinite $T_n \subseteq S_n$. TWO wins if $\bigcup_n T_n$ converges to x. Otherwise, ONE wins.

Proposition 2.10. *Assume that for each $x \in X$, ONE does not have a winning strategy in $\alpha_2^{\mathrm{game}}(X, x)$. Then X is α_{2^-} (and thus locally Ramsey).*

Proof. Assume that $\lim_m x_{nm} = x$ for all n. Consider the following strategy for ONE: In the first inning, ONE proposes $\{x_{1m} : m \in \mathbb{N}\}$. If TWO responds by

$$\{x_{1m} : m \in I_1\},$$

then ONE responds by

$$\{x_{2m} : m \in I_1 \setminus \{\min I_1\}\}.$$

In general, if in the nth inning TWO chooses a subsequence

$$\{x_{nm} : m \in I_n\},$$

then ONE responds by

$$\{x_{n+1,m} : m \in I_n \setminus \{\min I_n\}\}.$$

Since this strategy is not winning for ONE, there is a play lost by ONE. Let $I_1, I_2, \dots$ be the infinite sets of sequence indices, which correspond to the moves of TWO in this play. Define $m_n = \min I_n$ for each n. Then, for each n,

$$\bigcup_{n\in\mathbb{N}} \{x_{1m_n}, \dots, x_{nm_n}\} \subseteq \bigcup_n T_n,$$

which converges to e. □

Corollary 2.11. *Let G be a topological group. If ONE does not have a winning strategy in $\alpha_2^{\mathrm{game}}(G, e)$, then G is Ramsey (indeed, α_{2^-}).* □

Proposition 2.12. *If X is α_1, then for each $x \in X$, ONE does not have a winning strategy in $\alpha_2^{\mathrm{game}}(X, x)$.*

Proof. Define the game $\alpha_1^{\mathrm{game}}(X, x)$ corresponding to the property α_1 (at x). This game is similar to $\alpha_2^{\mathrm{game}}(X, x)$, with the only difference that here, TWO must choose a *cofinite* subset of each sequence provided by ONE.

Lemma 2.13. *X is α_1 if, and only if, for each $x \in X$, ONE does not have a winning strategy in $\alpha_1^{\mathrm{game}}(X, x)$.*

Proof. ($\Leftarrow$) Immediate.

($\Rightarrow$) The following trick was used by Scheepers in [13] to prove similar results for games involving open covers.

Fix a strategy for ONE in $\alpha_1^{\mathrm{game}}(X, x)$. For each sequence proposed by ONE, there are only countably many possible legal responds by TWO. Let $\mathcal{F}$ be the family of all possible sequences which ONE may propose in a play according to the fixed strategy. As $\mathcal{F}$ is countable, we can apply α_1 to $\mathcal{F}$, and find for each $S \in \mathcal{F}$ a cofinite subset $S' \subseteq S$, such that $\bigcup_{S\in\mathcal{F}} S'$ converges to x.

Now consider a play, where TWO responds to each given S_n by S'_n. This play is lost by ONE. □

Now, if ONE does not have a winning strategy in $\alpha_1^{\mathrm{game}}(X, x)$, in particular he does not have one in $\alpha_2^{\mathrm{game}}(X, x)$, where the moves of TWO are less restricted. □

To summarize, we have that, for topological groups,

$$\alpha_1 \Leftrightarrow \alpha_{1.5} \Rightarrow \text{ONE} \not\uparrow \alpha_2^{\mathrm{game}}(G, e) \Rightarrow \alpha_{2^-} \Rightarrow \text{Ramsey} \Rightarrow \alpha_{3^-} \Rightarrow \alpha_3 \Rightarrow \alpha_4$$

and

$$\alpha_{2^-} \Rightarrow \alpha_2 \Rightarrow \alpha_3.$$

We plan to include in the next version of this paper examples of topological groups separating some of the mentioned properties.

3. Sheaf amalgamations in topological vector spaces

Averbukh and Smolyanov asked wether for TVS's, α_1 implies Fréchet-Urysohn. This problem was implicitly solved in the field of Selection Principles, and explicitly by Plichko, using other methods.[1]

In the proof of the forthcoming Theorem 3.4, we will use several known facts, to which we provide proofs, for completeness.

For a product $\prod_{i\in I} X_i$ and $J\subset I$, $\mathrm{pr}_J : \prod_{i\in I} X_i \to \prod_{i\in J} X_i$ is projection on the coordinates in J, that is, restriction to J.

Lemma 3.1 (folklore)**.** *Assume that X is a hereditarily Lindelöf subspace of a product space $\prod_{i\in I} X_i$, Y is a second countable Hausdorff space, and $f : X\to Y$ is continuous. Then there are a countable $J\subseteq I$ and a continuous $g : \prod_{i\in J} X_i \to Y$ such that $f = g\circ \mathrm{pr}_J$.*

Proof. Let $\mathcal{B}$ be a countable base of the topology of Y. For every $B\in\mathcal{B}$ find a countable family $\mathcal{U}_B$ of standard basic open sets in $\prod_{i\in I} X_i$ such that $\bigcup\mathcal{U}_B\cap X = f^{-1}(B)\cap X$. Let J be the union of the supports of all $U\in\mathcal{U}_B$, $B\in\mathcal{B}$. J is countable. As Y is Hausdorff, for all $x_0,x_1\in X$ with $f(x_0)\neq f(x_1)$, $\mathrm{pr}_J(x_0)\neq \mathrm{pr}_J(x_1)$. Thus, we can define $g : \prod_{i\in J} X_i\to Y$ by $g(\mathrm{pr}_J(x)) = f(x)$ for all $x\in X$. By the choice of J, g is continuous. □

General versions of the following fact were proved in the seventies (e.g., [6] and references therein). Recall that the Σ-*product* of spaces X_i, $i\in I$, with respect to a point $x\in\prod_{i\in I} X_i$, is the subspace $\Sigma_{i\in I}X_i$ of the product space $\prod_{i\in I} X_i$, consisting of all $y\in\prod_{i\in I} X_i$ such that $y_i = x_i$ for all but countably many $i\in I$. (Each $y\in\prod_{i\in I} X_i$ can serve the role of x.)

Proposition 3.2. *Let X be a Σ-product of a family of first countable spaces. Then:*

1. *Each countable subspace of X is first countable.*
2. *X is α_1.*
3. *X has countable tightness.*
4. *X is Fréchet-Urysohn.*

Proof. (1) Countable subspaces of X are supported on a countable set of indices.

(2) Follows from (1).

(3) Let $X = \Sigma_{i\in I}X_i$, $A\subseteq X$ and $y\in\overline{A}$. For each $i\in I$, let $\mathcal{B}_i$ be a countable base at y_i. For finite $F\subseteq I$ and $U\in\prod_{i\in F}\mathcal{B}_i$, let

$$[U] = \{x\in X : \forall i\in F,\ x_i\in U_i\}.$$

Fix an arbitrary countably infinite $I_1\subseteq I$. Continue by induction on n. Let $A_n\subseteq A$ be a countable set intersecting $[U]$ for all finite $F\subseteq I_n$ and all $U\in\prod_{i\in F}\mathcal{B}_i$. Let I_{n+1} be the union of I_n and the supports of the elements of A_n.

y is in the closure of the countable set $\bigcup_n A_n$. Indeed, let F be a finite subset of I, and $U\in\prod_{i\in F}\mathcal{B}_i$. Let $F_1 = F\cap\bigcup_n I_n$, and $F_2 = F\setminus\bigcup_n I_n$. As F is finite, there is n such that $F_1\subseteq I_n$. Let $V = (U_i : i\in F_1)$. Then there is $a\in A_n\cap[V]$. As the support of a is contained in I_{n+1}, $a_i = y_i$ for all $i\in F_2$. Thus, $a\in[U]$.

(4) Follows from (3) and (1). □

[1] To be added: details, and the case of $C_p(X)$ for $X\subseteq\mathbb{R}$ (i.e. any non-γ-set bounded Borel images, e.g., under $\mathfrak{p}<\mathfrak{b}$ or CH.)

The following result, brought to our attention by Moore, is proved for S in [19, Theorem 7.10], where we are told that the L case is analogous. For completeness, we provide a proof for the L case, which is the one needed here.

Lemma 3.3. *Assume that Y is a regular topological space with all finite powers Lindelöf and countably tight, and $X \subseteq Y$ is non-separable. There exists a c.c.c. poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, X has an uncountable discrete subspace.*

Proof. It suffices to show that there are a c.c.c. poset $\mathbb{P}$ and a family $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{P}$, such that:

> For each model of ZFC $V' \supseteq V$ with $\omega_1^{V'} = \omega_1^V$, if there is in V' a filter $G \subseteq \mathbb{P}$ meeting each $D \in \mathcal{D}$, then X has an uncountable discrete subspace in V'.

Passing to a subset of X, if necessary, we may assume that $X = \{x_\xi : \xi < \omega_1\}$ and $\overline{\{x_\xi : \xi < \alpha\}}^Y \cap \{x_\eta : \eta \geq \alpha\} = \emptyset$ for every $\alpha < \omega_1$. We consider two cases.

Case 1. $\overline{\{x_\xi : \xi < \alpha\}}^Y \cap \overline{\{x_\eta : \eta \geq \alpha\}}^Y = \emptyset$ for all $\alpha < \omega_1$. (In other words, X is a free sequence in Y). Since Y has countable tightness, $\overline{X}^Y = \bigcup_{\alpha<\omega_1} \overline{\{x_\beta : \beta < \alpha\}}^Y$. The space $\overline{X}^Y$ is Lindelöf being a closed subspace of Y. On the other hand, it follows from the above that

$$\{\overline{X}^Y \setminus \overline{\{x_\beta : \beta \geq \alpha\}}^Y \ : \ \alpha < \omega_1\}$$

is an open cover of $\overline{X}^Y$ without a countable subcover, which leads to a contradiction. Thus case 1 is impossible.

Case 2. $\overline{\{x_\xi : \xi < \alpha\}}^Y \cap \overline{\{x_\eta : \eta \geq \alpha\}}^Y \neq \emptyset$ for some α. In particular, $\{x_\eta : \eta \geq \alpha\}$ is not compact. Without loss of generality we may assume that X is not compact. Let $\mathcal{U}$ be an ultrafilter on X whose elements are uncountable. If there exists some α such that $\mathcal{U}$ contains all open neighbourhoods in X of x_α, then the Hausdorff property implies that every x_β for $\beta \neq \alpha$ has a neighbourhood in X which is not in $\mathcal{U}$. Thus passing to a cofinal subset of X, if necessary, we may assume that every element of X has a neighbourhood in X which is not in $\mathcal{U}$. For every α let us find open neighbourhoods U_α, V_α of x_α in Y such that $\overline{V_\alpha} \subseteq U_\alpha$, $\overline{U_\alpha} \cap \overline{\{x_\xi : \xi < \alpha\}}^Y = \emptyset$, and $\{U_\alpha \cap X : \alpha < \omega_1\} \subseteq \mathcal{P}(X) \setminus \mathcal{U}$ (and hence finitely many of U_α's cannot cover a co-countable subset of X). Let $\mathbb{P}$ be the poset consisting of all finite sets $F \subseteq \omega_1$ such that, letting $n = |F|$ and $\{\alpha_0, \ldots, \alpha_{n-1}\}$ being the increasing enumeration of F, we have $x_{\alpha_j} \notin V_{\alpha_i}$ for all $i < j$. A condition H is stronger than F (in this case we write $H \leq F$) if $F \subseteq H$.

Assume, towards a contradiction, that there is an uncountable antichain $\{F_\alpha : \alpha < \omega_1\}$ in $\mathbb{P}$. Using the Δ-System Lemma and the fact that if $F, H \in \mathbb{P}$ are incompatible then so are $F \setminus H$ and $H \setminus F$, we may assume that F_α's are pairwise disjoint, $\min F_\alpha > \max F_\beta$ for all $\beta < \alpha$, and $|F_\alpha| = n$ for all α. Let $\{\xi_\alpha^0, \ldots, \xi_\alpha^{n-1}\}$ be the increasing enumeration of F_α. Set

$$W_\alpha^0 = \{(x_0, \ldots, x_{n-1}) \in X^n : \forall i, j < n\, (x_i \notin U_{\xi_\alpha^j})\},$$
$$W_\alpha^1 = \{(x_0, \ldots, x_{n-1}) \in X^n : \exists i, j < n\, (x_i \in \overline{V_{\xi_\alpha^j}})\}.$$

It is clear that $W_\alpha^0 \cap W_\alpha^1 = \emptyset$ and W_α^0, W_α^1 are closed. Moreover, $(x_{\xi_\beta^i})_{i<n} \in W_\alpha^0$ for all $\beta < \alpha$ by our choice of U_δ's, while $(x_{\xi_\beta^i})_{i<n} \in W_\alpha^1$ for all $\beta > \alpha$ by the

definition of $\mathbb{P}$ and the incompatibility of F_α and F_β. Therefore the subset $A = \{\vec{x}_\alpha = (x_{\xi^i_\alpha})_{i<n} : \alpha < \omega_1\}$ of X^n has the following property:

$$\overline{\{\vec{x}_\beta : \beta < \alpha\}}^{Y^n} \cap \overline{\{\vec{x}_\beta : \beta \geq \alpha\}}^{Y^n} = \emptyset.$$

This leads to a contradiction in the same way as in the first item. Thus $\mathbb{P}$ is c.c.c.

For every $\alpha < \omega_1$ consider the set $D_\alpha = \{F \in \mathbb{P} : \max F > \alpha\}$. Since no finite subfamily of $\{U_\alpha : \alpha < \omega_1\}$ covers a co-countable subset of X, each D_α is dense in $\mathbb{P}$. Now assume that G is a subfilter of $\mathbb{P}$ (maybe in some extension $V' \supseteq V$) which meets every D_α. Then $\bigcup G$ has the property that $x_\beta \notin V_\alpha$ for all $\beta, \alpha \in \bigcup G$ (if $\beta < \alpha$ this follows from the choice of V_α, while for $\beta > \alpha$ this follows from the fact that there must exist $F \in G$ containing both α and β). Therefore G gives rise to the discrete subspace $\{x_\alpha : \alpha \in \bigcup G\}$ of X, which is uncountable provided that $\omega_1^{V'} = \omega_1^V$. $\square$

We are now ready to prove the main result of this section.

An *L-space* is a hereditarily Lindelöf nonseparable topological space. The existence of L-spaces was established by Moore in his ground-breaking paper [7].

Theorem 3.4. *There is a hereditarily Lindelöf nonseparable Fréchet-Urysohn space L, such that:*

1. *Every countable subspace of $C_p(L)$ is metrizable. In particular, $C_p(L)$ is α_1.*
2. *$C_p(L)$ has uncountable tightness. In particular, $C_p(L)$ is not Fréchet-Urysohn.*

Proof. L is Moore's L-space [7]. Following Todorčević [18], Moore considered a function $\mathrm{osc} : \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\} \to \omega$, which has strong combinatorial properties. Let $(z_\alpha)_{\alpha<\omega_1}$ be a sequence of rationally independent points on the multiplicative circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For each $\beta < \omega_1$, define $w_\beta \in \mathbb{T}^{\omega_1}$ by

$$w_\beta(\alpha) = \begin{cases} z_\alpha^{\mathrm{osc}(\alpha,\beta)+1} & \alpha < \beta \\ 1 & \text{otherwise} \end{cases}$$

By Theorem 7.11 of [7], $L = \{w_\beta : \beta < \omega_1\}$ is an L-space. L is also Fréchet-Urysohn, either by Theorem 7.8 of [7], or directly by Proposition 3.2.

(1) Let D be a countable subset of $C_p(L)$.

By Lemma 3.1, for each continuous $f : L \to \mathbb{R}$, there are $\alpha < \omega_1$ and a continuous $g_\alpha : \mathrm{pr}_\alpha[L] \to \mathbb{R}$ such that $f = g_\alpha \circ \mathrm{pr}_\alpha$.

Lemma 3.5. *For each $\alpha < \omega_1$, $\mathrm{pr}_\alpha[L]$ is countable.*

Proof. By [7, Proposition 7.13], the subtree $\{\mathrm{osc}(\cdot, \delta) \restriction \alpha : \delta \geq \alpha\}$ of $\omega^{<\omega_1}$ is Aronszajn, where $\mathrm{osc}(\cdot, \delta) : \xi \mapsto \mathrm{osc}(\xi, \delta)$ for all $\xi < \delta$. By the definition of Aronszajn tree, we have that the set

$$\{\mathrm{osc}(\cdot, \delta) \restriction \alpha : \alpha < \delta < \omega_1\}$$

is countable for each $\alpha < \omega_1$. Thus, the set $\{w_\delta \restriction \alpha : \alpha < \delta < \omega_1\}$ is countable, and hence so is the set $\mathrm{pr}_\alpha[L]$. $\square$

As D is countable, there is $\alpha < \omega_1$ such that Lemma 3.1 holds for $J = \alpha$ and each $f \in D$. Thus, the function

$$\begin{array}{rcl} \mathrm{pr}_\alpha^* : C_p(\mathrm{pr}_\alpha[L]) & \to & C_p(L) \\ g & \mapsto & g \circ \mathrm{pr}_\alpha \end{array}$$

is an embedding (e.g., [3, Proposition 0.4.6]). As $\mathrm{pr}_\alpha[L]$ is countable, $C_p(\mathrm{pr}_\alpha[L])$ is metrizable, and therefore so is its image, which contains D.

(2) By Lemma 3.2, every finite power of $\Sigma_{\alpha<\omega_1}\mathbb{T}$ is countably tight. As countable tightness is hereditary, all finite powers of L are countably tight.

By Lemma 3.3 to $X = Y = L$, we conclude that if all finite powers of L are Lindelöf, then there is a c.c.c. poset $\mathbb{P}$ such that L has an uncountable discrete subspace in $V^{\mathbb{P}}$. But in the proof of [7, Theorem 7.17], we are told that L remains Lindelöf in c.c.c. forcing extension. Indeed, we have the following.

Lemma 3.6. *Moore's L-space remains an L-space in each forcing extension which does not collapse ω_1.*

Proof sketch. In accordance with [7, Definition 2.1], the construction of L is based on a C-sequence

$$\bar{C} = \langle C_\alpha : \alpha < \omega_1, \alpha \text{ limit}\rangle.$$

The function osc is constructed from $\bar{C}$ in a way that, for each poset $\mathbb{P}$ preserving ω_1, the constructions of osc in V and in $V^{\mathbb{P}}$ give the same function, and hence give rise to the same subspace of the Σ product of circles. By the same proof carried out in $V^{\mathbb{P}}$, this space is an L space in $V^{\mathbb{P}}$. □

It follows that some finite power of L is not Lindelöf. A classical result of Arhangel'skiĭ and, independently, Pytkeev, asserts that $C_p(X)$ has countable tightness if and only if all finite powers of X are Lindelöf. □

Problem 3.7.

(1) *Is the square of Moore's L-space non-Lindelöf?*
(2) *Is there, in ZFC, an L-space L such that L^2 is not Lindelöf?*

The prevalent opinion seems to be that the answer (to both questions) should be positive. Moore [7, Theorem 7.12] proved that the square L^2 of his L-space is not *hereditarily* Lindelöf.

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